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$p_I^\star$-open sets in ideal spaces*

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Abstract

The main idea of this paper is to present the collection of $p_I^\star$-open sets in ideal spaces. The relationships, characterizations and main properties of $p_I^\star$-open sets are obtained.

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1 Preliminaries and introduction

In topology, various collections of sets are used to get new characterizations of special spaces, basic maps, separation axioms, etc., for example: extremally disconnected space, hyperconnected spaces, locally indiscrete spaces, strongly Lindelöf spaces, etc. (for example [2, 3, 4, 13, 15, 17]). After initiation of the ideal spaces, valuable and new collections of sets were introduced to get new characterizations in the literature. These new collections of sets in ideal spaces were discussed for new characterizations, new properties in topology, for example [6, 8, 14]. In the present paper, the collection of $p_I^\star$-open subsets of ideal spaces is presented. The relationships, characterizations and main properties of $p_I^\star$-open sets are investigated.

Throughout the paper, (Y, ρ) denotes a topological space and $Cl(G)$ and $Int(G)$ denote the closure and the interior of a set G in Y, respectively.

Let $\mathbb{I}$ be a nonempty collection of subsets of a set Y. Then $\mathbb{I}$ is said to be an ideal on Y [19] if the following two properties are satisfied:

(1) If $H \in \mathbb{I}$ and $G \subset H$, we have $G \in \mathbb{I}$,
(2) If $G \in \mathbb{I}$ and $H \in \mathbb{I}$, we have $G \cup H \in \mathbb{I}$.

Definition 1.1. ([19]) Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y. Then $(.)^\star : P(Y) \longrightarrow P(Y)$, $G^\star = \{r \in Y : G \cap H \notin \mathbb{I}$ for each $H \in \rho$ such that $r \in H\}$ is said to be the local function of G with respect to $\mathbb{I}$ and ρ.

It is known that $Cl^\star(G) = G \cup G^\star$ is a Kuratowski closure operator and it will be denoted by $\rho^\star$ generated by $Cl^\star$ which will be called $\star$-topology [18].

Definition 1.2. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then G is called
(1) pre-I-open [5] if $G \subset Int(Cl^\star(G))$.
(2) semi-I-open [16] if $G \subset Cl^\star(Int(G))$.

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The complement of a semi-I-open set in a space with an ideal is called semi-I-closed.

Definition 1.3. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then G is called

(1) pre$^*_I$-open [11] if $G \subset Int^\star(Cl(G))$.

(2) pre$^*_I$-closed [7, 11] if $Y \setminus G$ is pre$^*_I$-open.

2 $p^\star_I$-open sets in ideal spaces

In this section, the collection of $p^\star_I$-open sets in ideal spaces is introduced. The relationships, characterizations of the collection of $p^\star_I$-open sets are discussed.

Definition 2.1. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then G is said to be a $p^\star_I$-open set if $G \in \{H \subset Y : H \neq \varnothing$ and there exists a nonempty $\star$-open set K such that $K \setminus Cl(H) \in \mathbb{I}\} \cup \{\varnothing\}$.

Theorem 2.2. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then the following properties are equivalent:

(1) G is a $p^\star_I$-open set in Y,

(2) $G = \varnothing$ or there exist a set O in $\mathbb{I}$ and a nonempty $\star$-open set N such that $N \setminus O \subset Cl(G)$.

Proof. (1) $\Rightarrow$ (2) : Let G be a $p^\star_I$-open set in Y. $G = \varnothing$ or $G \neq \varnothing$. Suppose $G \neq \varnothing$. There exists a nonempty $\star$-open set N such that $N \setminus Cl(G) \in \mathbb{I}$. Put $O = N \setminus Cl(G)$. This implies $O \in \mathbb{I}$ and $N \setminus O \subset Cl(G)$.

(2) $\Rightarrow$ (1) : Suppose that $G = \varnothing$ or there exist a set O in $\mathbb{I}$ and a nonempty $\star$-open set N such that $N \setminus O \subset Cl(G)$. If $G = \varnothing$, then G is a $p^\star_I$-open set in Y. Assume that there exist a set O in $\mathbb{I}$ and a nonempty $\star$-open set N such that $N \setminus O \subset Cl(G)$. We have $N \setminus Cl(G) \subset O$. This implies $N \setminus Cl(G) \in \mathbb{I}$. Consequently, G is a $p^\star_I$-open set in Y. ∎

Theorem 2.3. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then the following properties are equivalent:

(1) G is $p^\star_I$-open,

(2) $G \in \{H \subset Y : H \neq \varnothing$ and there exist a nonempty $\star$-open set O and a $F \in \mathbb{I}$ such that $O \subset Cl(H) \cup F\} \cup \{\varnothing\}$.

Proof. The proof follows from Theorem 2.2. ∎

Theorem 2.4. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. If G is a pre$^*_I$-open set in Y, then G is a $p^\star_I$-open set in Y.

Proof. Let G be a pre$^*_I$-open set in Y. If $G = \varnothing$, then G is a $p^\star_I$-open set in Y. Suppose $G \neq \varnothing$. We have $G \subset Int^\star(Cl(G))$. Put $O = Int^\star(Cl(G))$. This implies $O \neq \varnothing$ and $O \setminus Cl(G) \in \mathbb{I}$. Thus, G is a $p^\star_I$-open set in Y. ∎

Corollary 2.5. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. If G is a pre-I-open set in Y, then G is a $p^\star_I$-open set.

Proof. The proof follows by Theorem 2.4 since each pre-I-open set is pre$^*_I$-open. ∎

Remark 2.6. Let (Y,ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then the following diagram holds for G:

$$\begin{array}{ccccc} \text{pre-}I\text{-open} & \longrightarrow & \text{pre}_I^*\text{-open} & \longrightarrow & p_I^\star\text{-open} \\ & & \uparrow & & \\ & & \star\text{-open} & & \end{array}$$

Remark 2.7. The following example shows that the reverse implications of this diagram are not true in general.

Example 2.8. Let $Y = \{r_1, r_2, r_3, r_4\}$, $\rho = \{Y, \{r_1\}, \{r_2, r_3\}, \{r_1, r_2, r_3\}, \varnothing\}$ with an ideal $\mathbb{I} = \{\varnothing, \{r_1\}, \{r_4\}, \{r_1, r_4\}\}$. Then $G = \{r_1, r_4\} \subset Y$ is a $p_I^\star$-open set, G is not a pre$_I^*$-open set. Meanwhile, $H = \{r_3, r_4\} \subset Y$ is a pre$_I^*$-open set, H is neither a pre-I-open set nor a $\star$-open set.

Definition 2.9. Let (Y,ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then G is called $p_I^\star$-closed if $Y \setminus G$ is a $p_I^\star$-open set.

Theorem 2.10. Let (Y,ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then the following properties are equivalent:

(1) G is a $p_I^\star$-closed set in Y,

(2) $G = Y$ or there exist an element O of $\mathbb{I}$ and a $\star$-closed set $K \neq Y$ such that $Int(G) \setminus O \subset K$,

(3) $G = Y$ or there exist a $\star$-closed set $K \neq Y$ such that $Int(G) \setminus K \in \mathbb{I}$.

Proof. (1) $\Rightarrow$ (2) : Let G be a $p_I^\star$-closed subset of Y. Then $G = Y$ or $G \neq Y$. Suppose $G \neq Y$. This implies that $Y \setminus G \neq \varnothing$ and $Y \setminus G$ is $p_I^\star$-open. Then there exists a nonempty $\star$-open set N such that $N \setminus Cl(Y \setminus G) \in \mathbb{I}$. Put $O = N \setminus Cl(Y \setminus G)$. We have $O \in \mathbb{I}$ and

$$N \subset (Y \setminus Int(G)) \cup O.$$

This implies that the intersection of $Y \setminus (Y \setminus Int(G))$ and $Y \setminus O$ is contained in $Y \setminus N$.
Put $K = Y \setminus N$. Then K is a $\star$-closed set and $K \neq Y$.
Hence,

$$Int(G) \cap (Y \setminus O)$$

is contained in K. Consequently, we have $Int(G) \setminus O \subset K$.

(2) $\Rightarrow$ (1) : Suppose that $G = Y$ or there exist an element O of $\mathbb{I}$ and a $\star$-closed set $K \neq Y$ such that $Int(G) \setminus O \subset K$. If $G = Y$, then G is a $p_I^\star$-closed set in Y. Assume that there exist an element O of $\mathbb{I}$ and a $\star$-closed set $K \neq Y$ such that $Int(G) \setminus O \subset K$. This implies

$$Y \setminus K \subset (Y \setminus Int(G)) \cup O.$$

Take $N = Y \setminus K$. Then N is a nonempty $\star$-open set and we have $N \subset (Y \setminus Int(G)) \cup O$. Furthermore, $N \subset Cl(Y \setminus G) \cup O$. By Theorem 2.3, $Y \setminus G$ is a $p_I^\star$-open set in Y. Thus, G is a $p_I^\star$-closed set in Y.

(2) $\Rightarrow$ (3) : Suppose that there exist an element O of $\mathbb{I}$ and a $\star$-closed set $K \neq Y$ such that $Int(G) \setminus O \subset K$. Then we have $Int(G) \setminus K \subset O$. Consequently, we have $Int(G) \setminus K \in \mathbb{I}$.

(3) $\Rightarrow$ (2) : Suppose that there exist a $\star$-closed set $K \neq Y$ such that $Int(G) \setminus K \in \mathbb{I}$. Take $O = Int(G) \setminus K$. Then O is an element of $\mathbb{I}$ and $Int(G) \setminus O \subset K$. ∎

3 Special spaces and the relationships

In this section, some relationships and properties of the collection of $p_I^\star$-open sets with special spaces are investigated.

Definition 3.1. ([1]) Let (Y,ρ) be a space with an ideal $\mathbb{I}$ on Y. Then Y is called an F^*-space if each open set in Y is $\star$-closed.

Theorem 3.2. Let (Y,ρ) be a space with an ideal $\mathbb{I}$ on Y. Let Y be an F^*-space. Then every set in Y is a $p_I^\star$-open set.

Proof. Let $G \subset Y$. Then $G = \varnothing$ or $G \neq \varnothing$. Suppose that $G \neq \varnothing$. Since Y is an F^*-space, then $Cl(G)$ is $\star$-open. Also, we have $Cl(G) \neq \varnothing$. Take $O = Cl(G)$. This implies $O \setminus Cl(G) \in \mathbb{I}$. Thus, G is a $p_I^\star$-open set. ∎

Let (Y,ρ) be a topological space. Recall that Y is called a locally indiscrete if each open set in Y is closed.

Corollary 3.3. Let (Y,ρ) be a space with an ideal $\mathbb{I}$ on Y. Let Y be a locally indiscrete space. Then every set in Y is a $p_I^\star$-open set.

Proof. The proof follows from Theorem 3.2 since each locally indiscrete space is an F^*-space. ∎

Theorem 3.4. Let (Y,ρ) be a space with an ideal $\mathbb{I}$ on Y. Suppose that $\mathbb{I}$ has a $\star$-open element $\{r\}$. Then every set in Y is a $p_I^\star$-open set.

Proof. Let $\{r\}$ be a $\star$-open element of $\mathbb{I}$. Let $G \subset Y$. Then $G = \varnothing$ or $G \neq \varnothing$. Suppose $G \neq \varnothing$. We have $\{r\} \setminus Cl(G) = \{r\}$ or $\{r\} \setminus Cl(G) = \varnothing$. Since $\{r\}$ is a $\star$-open element of $\mathbb{I}$, then

$$\{r\} \setminus Cl(G) = \{r\} \in \mathbb{I} \text{ or } \{r\} \setminus Cl(G) = \varnothing \in \mathbb{I}.$$

Consequently, G is a $p_I^\star$-open set. ∎

Theorem 3.5. Let (Y,ρ) be a space with an ideal $\mathbb{I}$ on Y. Then every dense set in Y is a $p_I^\star$-open set.

Proof. Since every dense set is a pre_I^*-open set, it follows from Remark 2.6. ∎

Definition 3.6. ([10]) Let (Y,ρ) and (Z,η) be spaces with the ideals $\mathbb{I}$ and $\mathbb{I}'$ on Y and Z, respectively. A function $f : Y \longrightarrow Z$ is said to be $\star$-open if the image of each $\star$-open set in Y is $\star$-open in Z.

Theorem 3.7. Let $f : Y \longrightarrow Z$ be a function where (Y,ρ) and (Z,η) are spaces with the ideals $\mathbb{I}$ and $f(\mathbb{I})$ on Y and Z, respectively and $f(\mathbb{I}) = \{f(I) : I \in \mathbb{I}\}$. Suppose that f is bijection, $\star$-open and continuous. Then $f(G)$ is a $p_I^\star$-open set for any $p_I^\star$-open set G in Y.

Proof. Let G be a $p_I^\star$-open set G in Y and the function f be bijection, $\star$-open and continuous. Suppose $G \neq \varnothing$. Then there exists a nonempty $\star$-open set O such that $O \setminus Cl(G) \in \mathbb{I}$.

This implies
$$f(O) \setminus f(Cl(G)) \in f(\mathbb{I}).$$
Thus, $f(O) \setminus Cl(f(G)) \in f(\mathbb{I})$ and hence $f(G)$ is a $p_I^\star$-open set in Z. ∎

Definition 3.8. ([12]) Let (Y, ρ) and (Z, η) be spaces with the ideals $\mathbb{I}$ and $\mathbb{I}'$ on Y and Z, respectively. A function $f : Y \longrightarrow Z$ is said to be $\star$-closed if the image of each $\star$-closed set in Y is $\star$-closed in Z.

Corollary 3.9. Let $f : Y \longrightarrow Z$ be a function where (Y, ρ) and (Z, η) are spaces with the ideals $\mathbb{I}$ and $f(\mathbb{I})$ on Y and Z, respectively and $f(\mathbb{I}) = \{f(I) : I \in \mathbb{I}\}$. Suppose that f is bijection, $\star$-closed and continuous. Then $f(G)$ is $p_I^\star$-open for any $p_I^\star$-open subset G of Y.

Proof. The proof follows by Theorem 3.7. ∎

Theorem 3.10. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $\varnothing \neq G \subset Y$. If G is not a nowhere dense set, then G is a $p_I^\star$-open subset of Y.

Proof. Let $G \neq \varnothing$ be not a nowhere dense set. Then $Int(Cl(G)) \neq \varnothing$. We have $Int(Cl(G)) \subset Cl(G)$. Put $O = Int(Cl(G))$. This implies that O is a nonempty $\star$-open set and $O \setminus Cl(G) \in \mathbb{I}$. Thus, G is a $p_I^\star$-open set in Y. ∎

Definition 3.11. ([9]) Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $G \subset Y$. Then G is said to be a $\star$-nowhere dense set if $Int(Cl^\star(G)) = \varnothing$.

Corollary 3.12. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $\varnothing \neq G \subset Y$. If G is not a $\star$-nowhere dense set, then G is a $p_I^\star$-open set in Y.

Proof. The proof follows from Theorem 3.10. ∎

Remark 3.13. For any space (Y, ρ) with an ideal $\mathbb{I}$ on Y, there exists a nonempty $p_I^\star$-open set which is nowhere dense.

Example 3.14. Let $Y = \{r_1, r_2, r_3, r_4\}$, $\rho = \{Y, \{r_1\}, \{r_1, r_2\}, \{r_3, r_4\}, \{r_1, r_3, r_4\}, \varnothing\}$ with the ideal $\mathbb{I} = \{\varnothing, \{r_1\}, \{r_4\}, \{r_1, r_4\}\}$. Then $G = \{r_2\}$ is a $p_I^\star$-open set and also a nowhere dense set in Y.

Theorem 3.15. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y. Suppose that G_i is a $p_I^\star$-open set in Y for $i \in I$. Then $\cup_{i \in I} G_i$ is a $p_I^\star$-open set in Y.

Proof. Suppose that $G_i = \varnothing$ for each $i \in I$. Then $\cup_{i \in I} G_i$ is a $p_I^\star$-open set in Y.

Suppose that $G_{i_0} \neq \varnothing$ for an $i_0 \in I$. Then there exists a nonempty $\star$-open set O such that $O \setminus Cl(G_{i_0}) \in \mathbb{I}$. We have $Cl(G_{i_0}) \subset Cl(\cup_{i \in I} G_i)$. This implies
$$O \setminus Cl(\cup_{i \in I} G_i) \subset O \setminus Cl(G_{i_0}) \in \mathbb{I}.$$
Thus, $O \setminus Cl(\cup_{i \in I} G_i) \in \mathbb{I}$. Consequently, $\cup_{i \in I} G_i$ is a $p_I^\star$-open set in Y. ∎

Remark 3.16. For any space (Y, ρ) with an ideal $\mathbb{I}$ on Y, as is illustrated by the following example, there exist $p_I^\star$-open sets G and H but $G \cap H$ need not be $p_I^\star$-open.

Example 3.17. Let $Y = \{r_1, r_2, r_3, r_4\}$, $\rho = \{Y, \{r_1\}, \{r_2\}, \{r_1, r_2\}, \{r_1, r_2, r_3\}, \{r_1, r_2, r_4\}, \varnothing\}$ with the ideal $\mathbb{I} = \{\varnothing, \{r_4\}\}$. Then $G = \{r_1, r_3\}$ and $H = \{r_2, r_3\}$ are $p_I^\star$-open sets but $G \cap H$ is not a $p_I^\star$-open set in Y.

Theorem 3.18. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y. If $G \neq \varnothing$ is a $p_I^\star$-open set and $G \subset H$, then H is a $p_I^\star$-open set in Y.

Proof. Let $G \neq \varnothing$ be a $p_I^\star$-open set and $G \subset H$. Since $G \neq \varnothing$ is a $p_I^\star$-open set, then there exists a nonempty $\star$-open set O such that $O \setminus Cl(G) \in \mathbb{I}$. Since $G \subset H$, then $O \setminus Cl(H) \subset O \setminus Cl(G) \in \mathbb{I}$. Thus, $O \setminus Cl(H) \in \mathbb{I}$ and hence H is a $p_I^\star$-open set in Y. ∎

Theorem 3.19. Let (Y, ρ) be a space with an ideal $\mathbb{I}$ on Y and $r \in Y$. Then $\{r\}$ is semi-I-closed or $\{r\}$ is a $p_I^\star$-open set in Y.

Proof. Assume that $\{r\}$ is not a semi-I-closed set in Y. This implies $Int^\star(Cl(\{r\})) \nsubseteq \{r\}$. We have $Int^\star(Cl(\{r\})) \setminus Cl(\{r\}) \in \mathbb{I}$. Hence, $\{r\}$ is a $p_I^\star$-open set in Y. ∎

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